

THE CLUSTER CHARACTER FOR CYCLIC QUIVERS

MING DING AND FAN XU

ABSTRACT. We define an analogue of the Caldero-Chapoton map ([CC]) for the cluster category of finite dimensional nilpotent representations over a cyclic quiver. We prove that it is a cluster character (in the sense of [Pa]) and satisfies some inductive formulas for the multiplication between the generalized cluster variables (the images of objects of the cluster category under the map). Moreover, we construct a \mathbb{Z} -basis for the algebras generated by all generalized cluster variables.

CONTENTS

1. Introduction	1
2. The cluster character for cyclic quivers	2
3. Inductive multiplication formulas	6
4. A \mathbb{Z} -basis for cyclic quivers	8
Acknowledgements	10
References	10

1. INTRODUCTION

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [FZ] in order to develop a combinatorial approach to study problems of total positivity in algebraic groups and canonical bases in quantum groups. The link between cluster algebras and representation theory of quivers were first revealed in [MRZ]. In [BMRRT], the authors introduced the cluster category of an acyclic quiver Q (a quiver without oriented cycles) as the categorification of the corresponding cluster algebras. In order to show that a cluster category categorifies the involving cluster algebra, the Caldero-Chapoton map was defined by P. Caldero and F. Chapoton in [CC]. Let $\mathcal{C}(Q)$ be the cluster category associated to an acyclic quiver Q (a quiver without oriented cycles). The Caldero-Chapoton map of an acyclic quiver Q is a map

$$X_{\tau}^Q : \text{obj}(\mathcal{C}(Q)) \rightarrow \mathbb{Q}(x_1, \dots, x_n).$$

The map was extensively defined by Y. Palu for a Hom-finite 2-Calabi-Yau triangulated categories with a cluster tilting object ([Pa]).

As in [Ke], the cluster category can be defined for any small hereditary abelian category with finite dimensional Hom- and Ext-spaces. It is interesting to study the cluster categories without cluster tilting objects and the involving cluster algebras. For example, the cluster category of a 1-Calabi-Yau abelian category contains no cluster tilting objects (even no rigid objects).

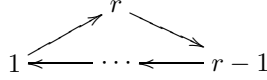
Key words and phrases: cyclic quiver, cluster algebra, \mathbb{Z} -basis.

Fan Xu was supported by Alexander von Humboldt Stiftung and was also partially supported by the Ph.D. Programs Foundation of Ministry of Education of China (No. 200800030058).

In this paper, we will focus on the simplest example of the cluster category without cluster tilting objects: the cluster category of a cyclic quiver. We first define an analogue of the Caldero-Chapoton map for a cyclic quiver. We prove a multiplication formula analogous to the cluster multiplication theorem for acyclic cluster algebras ([CK], [XX]). As a corollary, the map is a cluster character in the sense of [Pa]. Let \tilde{A}_r be the cyclic quiver with r vertices and $\mathcal{C}(\tilde{A}_r)$ be its cluster category. Let \mathcal{AH} be the subalgebra of $\mathbb{Q}(x_1, \dots, x_r)$ generated by $\{X_M \mid M \in \text{mod } \mathbb{C}\tilde{A}_r\}$ and \mathcal{EH} be the subalgebra of \mathcal{AH} generated by $\{X_M \mid M \in \text{mod } \mathbb{C}\tilde{A}_r, \text{Ext}_{\mathcal{C}(\tilde{A}_r)}^1(M, M) = 0\}$. We call the algebra \mathcal{EH} the cluster algebra of \tilde{A}_r . We show that \mathcal{AH} coincides with \mathcal{EH} and construct a \mathbb{Z} -basis of \mathcal{EH} .

2. THE CLUSTER CHARACTER FOR CYCLIC QUIVERS

Let $k = \mathbb{C}$ be the field of complex number. Throughout the rest part of this paper, we fix a cyclic quiver $Q = \tilde{A}_r$, i.e., a quiver with oriented cycles, where $Q_0 = \{1, 2, \dots, r\}$:



We denote by $\text{mod } kQ$ the category of finite-dimensional nilpotent representations of kQ . Let τ be the Auslander-Reiten translation functor. Let E_1, \dots, E_r be simple modules of the vertices $1, \dots, r$, respectively. Set $\underline{\dim} E_i = s_i$ for $i = 1, \dots, r$. We have $\tau E_2 = E_1, \dots, \tau E_1 = E_r$. The Auslander-Reiten quiver of kQ is a tube of rank r with E_1, \dots, E_r lying the mouth of the tube. For $1 \leq i \leq r$, we denote by $E_i = E_{i+r}$, and by $E_i[n]$ the unique nilpotent indecomposable representation with socle E_i and length n . Set $E_i[0] = 0$ for $i = 1, \dots, r$. We note that any indecomposable kQ -module is of the form $E_i[j]$ for $i = 1, \dots, r$ and $j \in \mathbb{N} \sqcup \{0\}$. Let $\mathbf{x} = \{x_i \mid i \in Q_0\}$ is a family indeterminates over \mathbb{Z} and set $x_i = x_{i+mr}$ for $1 \leq i \leq r, m \in \mathbb{Z}_{\geq 0}$. Here we denote by $\mathbb{Z}_{\geq 0} = \mathbb{Z} \sqcup \{0\}$.

By definition, the cluster category $\mathcal{C} = \mathcal{C}(Q)$ is the orbit category $\mathcal{D}^b(\text{mod } kQ)/\tau \circ [-1]$. It is a triangulated category by [Ke, Theorem 9.9]. Different from the cluster category of an acyclic quiver, the set of objects in \mathcal{C} coincides with the set of objects in $\text{mod } kQ$. Also, for any two indecomposable objects $M, N \in \mathcal{C}$, we have

$$\text{Ext}_{\mathcal{C}}^1(M, N) = \text{Ext}_{kQ}^1(M, N) \oplus \text{Ext}_{kQ}^1(N, M).$$

It is possible that both of two terms in the right side don't vanish. We denote by $\langle -, - \rangle$ the Euler form on $\text{mod } kQ$, i.e., for any M, N in $\text{mod } kQ$,

$$\langle \underline{\dim} M, \underline{\dim} N \rangle := \dim_k \text{Hom}_{kQ}(M, N) - \dim_k \text{Ext}_{kQ}^1(M, N).$$

It is well defined. Thus according to the Caldero-Chapoton map, we can similarly define a map on $\text{mod } kQ$

$$X_{\bullet} : \text{obj}(\text{mod } kQ) \longrightarrow \mathbb{Z}[\mathbf{x}^{\pm 1}]$$

by mapping M to

$$X_M = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \underline{e}, s_i \rangle - \langle s_i, \underline{\dim} M - \underline{e} \rangle}$$

where $\text{Gr}_{\underline{e}}(M)$ is the \underline{e} -Grassmannian of M , i.e., the variety of finite-dimensional nilpotent submodules of M with dimension vector \underline{e} , and set $X_0 = 1$. Here, we need not assume that M is indecomposable by the following Proposition 2.2. The fraction X_M is called a generalized cluster variable.

Proposition 2.1. *With the above notation, we have*

$$X_{E_l[n]} = \frac{x_{l+n}}{x_l} + \sum_{k=1}^{n-1} \frac{x_{l+n}x_{l+r-1}}{x_{l+k-1}x_{l+k}} + \frac{x_{l+r-1}}{x_{l+n-1}}.$$

for $n \in \mathbb{N}$ and $l = 1, \dots, r$.

Proof. It is known that all submodules of $E_l[n]$ are $E_l[0], E_l[1], \dots, E_l[n]$. Set $\underline{d}_{i,j} = \underline{\dim} E_i[j]$. By definition,

$$X_{E_l[n]} = \sum_{k=0}^n \prod_{i \in Q_0} x_i^{-\langle \underline{d}_{l,k}, \underline{d}_{i,1} \rangle - \langle \underline{d}_{i,1}, \underline{d}_{l+k, n-k} \rangle}.$$

By the definition of the Euler form, we have

$$\begin{aligned} -\langle \underline{d}_{l,k}, \underline{d}_{i,1} \rangle - \langle \underline{d}_{i,1}, \underline{d}_{l+k, n-k} \rangle &= -\dim_k \operatorname{Hom}(E_l[k], E_i) + \dim_k \operatorname{Ext}^1(E_l[k], E_i) \\ &\quad - \dim_k \operatorname{Hom}(E_i, E_{l+k}[n-k]) + \dim_k \operatorname{Ext}^1(E_i, E_{l+k}[n-k]). \end{aligned}$$

If $k = 0$, then $\prod_{i \in Q_0} x_i^{-\langle \underline{d}_{l,k}, \underline{d}_{i,1} \rangle - \langle \underline{d}_{i,1}, \underline{d}_{l+k, n-k} \rangle} = \frac{x_{l+n}}{x_l}$.

If $k = n$, then $\prod_{i \in Q_0} x_i^{-\langle \underline{d}_{l,k}, \underline{d}_{i,1} \rangle - \langle \underline{d}_{i,1}, \underline{d}_{l+k, n-k} \rangle} = \frac{x_{l+r-1}}{x_{l+n-1}}$.

If $0 < k < n$, then $\prod_{i \in Q_0} x_i^{-\langle \underline{d}_{l,k}, \underline{d}_{i,1} \rangle - \langle \underline{d}_{i,1}, \underline{d}_{l+k, n-k} \rangle} = \frac{x_{l+n}x_{l+r-1}}{x_{l+k-1}x_{l+k}}$. \square

Proposition 2.2. (1) *For M, N in $\operatorname{mod} kQ$, we have*

$$X_M X_N = X_{M \oplus N}.$$

(2) *Let $0 \longrightarrow \tau M \longrightarrow B \longrightarrow M \longrightarrow 0$ be an almost split sequence in $\operatorname{mod} kQ$, then*

$$X_M X_{\tau M} = X_B + 1.$$

Proof. The proof is similar to [CC, Proposition 3.6]. For (1), by definition, it is enough to prove that for any dimension vector \underline{e} , we have

$$\chi(\operatorname{Gr}_{\underline{e}}(M \oplus N)) = \sum_{\underline{f} + \underline{g} = \underline{e}} \chi(\operatorname{Gr}_{\underline{f}}(M)) \chi(\operatorname{Gr}_{\underline{g}}(N)).$$

Consider the natural morphism of varieties

$$f : \prod_{\underline{f} + \underline{g} = \underline{e}} \operatorname{Gr}_{\underline{f}}(M) \times \operatorname{Gr}_{\underline{g}}(N) \rightarrow \operatorname{Gr}_{\underline{e}}(M \oplus N)$$

defined by sending (M_1, N_1) to $M_1 \oplus N_1$. Since f is monomorphism, we have

$$\sum_{\underline{f} + \underline{g} = \underline{e}} \chi(\operatorname{Gr}_{\underline{f}}(M)) \chi(\operatorname{Gr}_{\underline{g}}(N)) = \chi(\operatorname{Im} f).$$

On the other hand, we define an action of \mathbb{C}^* on $\operatorname{Gr}_{\underline{e}}(M \oplus N)$ by

$$t.(m, n) = (tm, t^2n)$$

for $t \in \mathbb{C}^*$ and $m \in M, n \in N$. The set of stable points is just $\operatorname{Im} f$. Hence, $\chi(\operatorname{Im} f) = \chi(\operatorname{Gr}_{\underline{e}}(M \oplus N))$. This proves (1).

(2) Assume that $M = E_i[n]$. It is enough to prove:

$$X_{E_1[n]} X_{E_2[n]} = X_{E_1[n+1]} X_{E_2[n-1]} + 1$$

where $0 \longrightarrow E_1[n] \longrightarrow E_2[n-1] \oplus E_1[n+1] \longrightarrow E_2[n] \longrightarrow 0$ is an almost split sequence in $\operatorname{mod} kQ$. The equation in (2) follows from the direct confirmation by using Proposition 2.1. \square

Let M, N be indecomposable kQ -modules satisfying that $\dim_k \text{Ext}_{kQ}^1(M, N) = \dim_k \text{Hom}_{kQ}(N, \tau M) = 1$. Assume that $M = E_i[j]$, $N = E_k[l]$. Then in $\mathcal{C}(Q)$, there are just two involving triangles

$$E_k[l] \rightarrow E \rightarrow E_i[j] \rightarrow \tau E_k[l]$$

and

$$E_i[j] \rightarrow E' \rightarrow E_k[l] \xrightarrow{g} \tau E_i[j]$$

where $E \cong E_k[i+j-k] \oplus E_i[k+l-i]$ and $E' \cong \ker g \oplus \tau^{-1} \text{coker } g$.

Theorem 2.3. *With the above notation, we have*

$$X_M X_N = X_E + X_{E'}.$$

Proof. Assume that

$$X_M = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}(M)) \prod_{i \in Q_0} x_i^{-\langle \underline{e}, s_i \rangle - \langle s_i, \underline{\dim} M - \underline{e} \rangle}$$

and

$$X_N = \sum_{\underline{e}'} \chi(\text{Gr}_{\underline{e}'}(N)) \prod_{i \in Q_0} x_i^{-\langle \underline{e}', s_i \rangle - \langle s_i, \underline{\dim} N - \underline{e}' \rangle}$$

Then

$$X_M X_N = \sum_{\underline{e}, \underline{e}'} \prod_{i \in Q_0} x_i^{-\langle \underline{e} + \underline{e}', s_i \rangle - \langle s_i, \underline{\dim} M + \underline{\dim} N - (\underline{e} + \underline{e}') \rangle}.$$

Note that $\chi(\text{Gr}_{\underline{e}}(L)) = 1$ or 0 for any kQ -module L . Since $\text{Ext}_{kQ}^1(M, N) \neq 0$, we have a short exact sequence

$$0 \rightarrow E_k[l] \xrightarrow{f_1} E \xrightarrow{f_2} E_i[j] \rightarrow 0.$$

Define a morphism of varieties

$$\phi : \text{Gr}_{\underline{d}}(E) \rightarrow \bigsqcup_{\underline{e} + \underline{e}' = \underline{d}} \text{Gr}_{\underline{e}}(M) \times \text{Gr}_{\underline{e}'}(N)$$

by sending (E_1) to $(f_1^{-1}(E_1), f_2(E_1))$. For $(M_1, N_1) \in \text{Gr}_{\underline{e}}(M) \times \text{Gr}_{\underline{e}'}(N)$, we consider the natural map:

$$\beta' : \text{Ext}_{kQ}^1(M, N) \oplus \text{Ext}_{kQ}^1(M_1, N_1) \rightarrow \text{Ext}_{kQ}^1(M_1, N)$$

sending $(\varepsilon, \varepsilon')$ to $\varepsilon_{M_1} - \varepsilon'_N$ where ε_{M_1} and ε'_N are induced by including $M_1 \subseteq M$ and $N_1 \subseteq N$, respectively and the projection

$$p_0 : \text{Ext}_{kQ}^1(M, N) \oplus \text{Ext}_{kQ}^1(M_1, N_1) \rightarrow \text{Ext}_{kQ}^1(M, N).$$

It is easy to check that $(M_1, N_1) \in \text{Im } \phi$ if and only if $p_0(\ker \beta') \neq 0$. Hence, we have

$$X_E = \sum_{\underline{e}, \underline{e}'; p_0(\ker \beta') \neq 0} \prod_{i \in Q_0} x_i^{-\langle \underline{e} + \underline{e}', s_i \rangle - \langle s_i, \underline{\dim} M + \underline{\dim} N - (\underline{e} + \underline{e}') \rangle}.$$

Assume that

$$X_{E'} = \sum_{\underline{d}'_1, \underline{d}'_2} \chi(\text{Gr}_{\underline{d}'_1}(K)) \chi(\text{Gr}_{\underline{d}'_2}(\tau^{-1}C)) \prod_{i \in Q_0} x_i^{-\langle \underline{d}'_1 + \underline{d}'_2, s_i \rangle - \langle s_i, \underline{\dim} K + \underline{\dim} \tau^{-1}C - \underline{d}'_1 - \underline{d}'_2 \rangle}$$

Set $\underline{d}^* = \underline{\dim} M - \underline{\dim} \tau^{-1}C$. We have

$$\begin{aligned} & \langle s_i, \underline{\dim} K + \underline{\dim} \tau^{-1}C \rangle \\ &= \langle s_i, \underline{\dim} N - \tau(\underline{d}^*) + \underline{\dim} M - \underline{d}^* \rangle \\ &= \langle s_i, \underline{\dim} M + \underline{\dim} N - \underline{d}^* \rangle + \langle \underline{d}^*, s_i \rangle. \end{aligned}$$

Hence, $X_{E'}$ can be reformulated as

$$\sum_{\underline{d}'_1, \underline{d}'_2} \prod_{i \in Q_0} x_i^{-\langle \underline{d}'_1 + \underline{d}'_2 + \underline{d}^*, s_i \rangle - \langle s_i, \underline{\dim} M + \underline{\dim} N - (\underline{d}'_1 + \underline{d}'_2 + \underline{d}^*) \rangle}$$

Since $\dim_k \text{Hom}_{kQ}(N, \tau M) = 1$, there is only one element in $\mathbb{P}\text{Hom}_{kQ}(N, \tau M)$ with the representative g . We have a long exact sequence

$$0 \longrightarrow K \longrightarrow N \xrightarrow{g} \tau M \longrightarrow C \longrightarrow 0$$

Given submodules K_1, C_1 of K, C , respectively, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & N & \xrightarrow{g} & \tau M & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K/K_1 & \longrightarrow & N/K_1 & \xrightarrow{g'} & \tau M_1 & \longrightarrow & C_1 & \longrightarrow & 0 \end{array}$$

where τM_1 is the corresponding pullback. Define a morphism of varieties

$$\phi' : \bigsqcup_{\underline{d}'_1 + \underline{d}'_2 + \underline{d}^* = \underline{d}'} Gr_{\underline{d}'_1}(K) \times Gr_{\underline{d}'_2}(\tau^{-1}C) \rightarrow \bigsqcup_{\underline{e} + \underline{e}' = \underline{d}'} Gr_{\underline{e}}(M) \times Gr_{\underline{e}'}(N)$$

by sending $(K_1, \tau^{-1}(C_1))$ to (K_1, M_1) . Checking the above diagram, we know that $(M_1, N_1) \in \text{Im} \phi'$ if and only if $\text{Hom}_{kQ}(N/N_1, \tau M_1) \neq 0$. Therefore, we obtain

$$X_{E'} = \sum_{\underline{e}, \underline{e}'; \text{Hom}_{kQ}(N/N_1, \tau M_1) \neq 0} \prod_{i \in Q_0} x_i^{-\langle \underline{e} + \underline{e}', s_i \rangle - \langle s_i, \underline{\dim} M + \underline{\dim} N - (\underline{e} + \underline{e}') \rangle}.$$

Consider the dual of β' :

$$\beta : \text{Hom}_{kQ}(N, \tau M_1) \rightarrow \text{Hom}_{kQ}(N, \tau M) \oplus \text{Hom}_{kQ}(N_1, \tau M_1).$$

Then

$$(p_0(\ker \beta'))^\perp = \text{Im} \beta \bigcap \text{Hom}(N, \tau M) \simeq \text{Hom}(N/N_1, \tau M_1).$$

We obtain that

$$\dim_k(p_0(\ker \beta')) + \dim_k \text{Hom}(N/N_1, \tau M_1) = \dim_k \text{Ext}_{kQ}^1(M, N) = 1$$

Hence, any (M_1, N_1) belongs to either $\text{Im} \phi$ or $\text{Im} \phi'$ for some \underline{d} or \underline{d}' . We complete the proof. \square

Following the definition of a cluster character in [Pa], we can easily check the following corollary.

Corollary 2.4. *The Caldero-Chapoton map for a cyclic quiver is a cluster character.*

We will construct some inductive formulas in the next section. For convenience, we write down the following corollary.

Corollary 2.5. *With the above notation, we have*

- (1) $X_{E_{i+n}} X_{E_i[n]} = X_{E_i[n+1]} + X_{E_i[n-1]}$
- (2) $X_{E_i} X_{E_{i+1}[n]} = X_{E_i[n+1]} + X_{E_{i+2}[n-1]}.$

3. INDUCTIVE MULTIPLICATION FORMULAS

In this section, we will give inductive multiplication formulas for any two generalized cluster variables on $\text{mod } kQ$. Note that these inductive multiplication formulas are an analogue of those for tubes in [DXX] for acyclic cluster algebras.

Theorem 3.1. *Let i, j, k, l, m and r be in \mathbb{Z} such that $1 \leq k \leq mr + l, 0 \leq l \leq r - 1, 1 \leq i, j \leq r, m \geq 0$.*

(1) *When $j \leq i$, then*

1) *for $k + i \geq r + j$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[(m+1)r+l+j-i]}X_{E_j[k+i-r-j]} + X_{E_i[r+j-i-1]}X_{E_{k+i+1}[(m+1)r+l+j-k-i-1]}$,*

2) *for $k + i < r + j$ and $i \leq l + j \leq k + i - 1$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_j[mr+k+i-j]}X_{E_i[l+j-i]} + X_{E_j[mr+i-j-1]}X_{E_{l+j+1}[k+i-l-j-1]}$,*

3) *for other conditions, i.e., there are no extension between $E_i[k]$ and $E_j[mr+l]$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[k] \oplus E_j[mr+l]}$.*

(2) *When $j > i$, then*

1) *for $k \geq j - i$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[j-i-1]}X_{E_{k+i+1}[mr+l+j-k-i-1]} + X_{E_i[mr+l+j-i]}X_{E_j[k+i-j]}$,*

2) *for $k < j - i$ and $i \leq l + j - r \leq k + i - 1$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_j[(m+1)r+k+i-j]}X_{E_i[l+j-r-i]} + X_{E_j[(m+1)r+i-j-1]}X_{E_{l+j+1}[k+r+i-l-j-1]}$,*

3) *for other conditions, i.e., there are no extension between $E_i[k]$ and $E_j[mr+l]$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[k] \oplus E_j[mr+l]}$.*

Proof. We only prove (1) and (2) is totally similar to (1).

1) When $k = 1$, by $k + i \geq r + j$ and $1 \leq j \leq i \leq r \implies i = r$ and $j = 1$.

Then by Proposition 2.2 and Corollary 2.5, we have

$$X_{E_r}X_{E_1[mr+l]} = X_{E_r[mr+l+1]} + X_{E_2[mr+l-1]}.$$

When $k = 2$, by $k + i \geq r + j$ and $1 \leq j \leq i \leq r \implies i = r$ or $i = r - 1$.

For $i = r \implies j = 1$ or $j = 2$:

The case for $i = r$ and $j = 1$, we have

$$\begin{aligned} & X_{E_r[2]}X_{E_1[mr+l]} \\ &= (X_{E_r}X_{E_1} - 1)X_{E_1[mr+l]} \\ &= X_{E_1}(X_{E_r[mr+l+1]} + X_{E_2[mr+l-1]}) - X_{E_1[mr+l]} \\ &= X_{E_1}X_{E_r[mr+l+1]} + (X_{E_1[mr+l]} + X_{E_3[mr+l-2]}) - X_{E_1[mr+l]} \\ &= X_{E_1}X_{E_r[mr+l+1]} + X_{E_3[mr+l-2]}. \end{aligned}$$

The case for $i = r$ and $j = 2$, we have

$$\begin{aligned} & X_{E_r[2]}X_{E_2[mr+l]} \\ &= (X_{E_r}X_{E_1} - 1)X_{E_2[mr+l]} \\ &= X_{E_r}(X_{E_1[mr+l+1]} + X_{E_3[mr+l-1]}) - X_{E_2[mr+l]} \\ &= X_{E_r[mr+l+2]} + (X_{E_2[mr+l]} + X_{E_r}X_{E_3[mr+l-1]}) - X_{E_2[mr+l]} \\ &= X_{E_r[mr+l+2]} + X_{E_r}X_{E_3[mr+l-1]}. \end{aligned}$$

For $i = r - 1 \implies j = 1$:

$$\begin{aligned} & X_{E_{r-1}[2]}X_{E_1[mr+l]} \\ &= (X_{E_{r-1}}X_{E_r} - 1)X_{E_1[mr+l]} \\ &= X_{E_{r-1}}(X_{E_r[mr+l+1]} + X_{E_2[mr+l-1]}) - X_{E_1[mr+l]} \\ &= (X_{E_{r-1}[mr+l+2]} + X_{E_1[mr+l]}) + X_{E_{r-1}}X_{E_2[mr+l-1]} - X_{E_1[mr+l]} \\ &= X_{E_{r-1}[mr+l+2]} + X_{E_{r-1}}X_{E_2[mr+l-1]}. \end{aligned}$$

Now, suppose it holds for $k \leq n$, then by induction we have

$$\begin{aligned}
& X_{E_i[n+1]}X_{E_j[mr+l]} \\
&= (X_{E_i[n]}X_{E_{i+n}} - X_{E_i[n-1]})X_{E_j[mr+l]} \\
&= X_{E_{i+n}}(X_{E_i[n]}X_{E_j[mr+l]}) - X_{E_i[n-1]}X_{E_j[mr+l]} \\
&= X_{E_{i+n}}(X_{E_i[(m+1)r+l+j-i]}X_{E_j[n+i-r-j]} + X_{E_i[r+j-i-1]}X_{E_{n+i+1}[(m+1)r+l+j-n-i-1]}) \\
&\quad - (X_{E_i[(m+1)r+l+j-i]}X_{E_j[n+i-r-j-1]} + X_{E_i[r+j-i-1]}X_{E_{n+i}[(m+1)r+l+j-n-i]}) \\
&= X_{E_i[(m+1)r+l+j-i]}(X_{E_j[n+i+1-r-j]} + X_{E_j[n+i-r-j-1]}) \\
&\quad + X_{E_i[r+j-i-1]}(X_{E_{n+i}[(m+1)r+l+j-n-i]} + X_{E_{n+i+2}[(m+1)r+l+j-n-i-2]}) \\
&\quad - (X_{E_i[(m+1)r+l+j-i]}X_{E_j[n+i-r-j-1]} + X_{E_i[r+j-i-1]}X_{E_{n+i}[(m+1)r+l+j-n-i]}) \\
&= X_{E_i[(m+1)r+l+j-i]}X_{E_j[n+i+1-r-j]} + X_{E_i[r+j-i-1]}X_{E_{n+i+2}[(m+1)r+l+j-n-i-2]}.
\end{aligned}$$

2) When $k = 1$, by $i \leq l+j \leq k+i-1 \implies i \leq l+j \leq i \implies i = l+j$.

Then by Proposition 2.2 and Corollary 2.5, we have

$$X_{E_i}X_{E_j[mr+l]} = X_{E_{l+j}}X_{E_j[mr+l]} = X_{E_j[mr+l+1]} + X_{E_j[mr+l-1]}$$

When $k = 2$, by $i \leq l+j \leq k+i-1 \implies i \leq l+j \leq i+1 \implies i = l+j$ or $i+1 = l+j$:

For $i = l+j$, we have

$$\begin{aligned}
& X_{E_i[2]}X_{E_j[mr+l]} \\
&= X_{E_{l+j}[2]}X_{E_j[mr+l]} \\
&= (X_{E_{l+j}}X_{E_{l+j+1}} - 1)X_{E_j[mr+l]} \\
&= (X_{E_j[mr+l+1]} + X_{E_j[mr+l-1]})X_{E_{l+j+1}} - X_{E_j[mr+l]} \\
&= X_{E_j[mr+l+2]} + X_{E_j[mr+l]} + X_{E_{l+j+1}}X_{E_j[mr+l-1]} - X_{E_j[mr+l]} \\
&= X_{E_j[mr+j+2]} + X_{E_{l+j+1}}X_{E_j[mr+l-1]}.
\end{aligned}$$

For $i+1 = l+j$, we have

$$\begin{aligned}
& X_{E_i[2]}X_{E_j[mr+l]} \\
&= X_{E_{l+j-1}[2]}X_{E_j[mr+l]} \\
&= (X_{E_{l+j-1}}X_{E_{l+j}} - 1)X_{E_j[mr+l]} \\
&= (X_{E_j[mr+l+1]} + X_{E_j[mr+l-1]})X_{E_{l+j-1}} - X_{E_j[mr+l]} \\
&= X_{E_j[mr+l+1]}X_{E_{l+j-1}} + (X_{E_j[mr+l]} + X_{E_j[mr+l-2]}) - X_{E_j[mr+l]} \\
&= X_{E_j[mr+l+1]}X_{E_{l+j-1}} + X_{E_j[mr+l-2]}.
\end{aligned}$$

Suppose it holds for $k \leq n$, then by induction we have

$$\begin{aligned}
& X_{E_i[n+1]}X_{E_j[mr+l]} \\
&= (X_{E_i[n]}X_{E_{i+n}} - X_{E_i[n-1]})X_{E_j[mr+l]} \\
&= (X_{E_i[n]}X_{E_j[mr+l]})X_{E_{i+n}} - X_{E_i[n-1]}X_{E_j[mr+l]} \\
&= (X_{E_j[mr+n+i-j]}X_{E_i[l+j-i]} + X_{E_j[mr+i-j-1]}X_{E_{l+j+1}[n+i-l-j-1]})X_{E_{i+n}} \\
&\quad - (X_{E_j[mr+n+i-j-1]}X_{E_i[l+j-i]} + X_{E_j[mr+i-j-1]}X_{E_{l+j+1}[n+i-l-j-2]}) \\
&= (X_{E_j[mr+n+i+1-j]} + X_{E_j[mr+n+i-j-1]})X_{E_i[l+j-i]} \\
&\quad + (X_{E_{l+j+1}[n+i-l-j]} + X_{E_{l+j+1}[n+i-l-j-2]})X_{E_j[mr+i-j-1]} \\
&\quad - (X_{E_j[mr+n+i-j-1]}X_{E_i[l+j-i]} + X_{E_j[mr+i-j-1]}X_{E_{l+j+1}[n+i-l-j-2]}) \\
&= X_{E_j[mr+n+i+1-j]}X_{E_i[l+j-i]} + X_{E_j[mr+i-j-1]}X_{E_{l+j+1}[n+i-l-j]}.
\end{aligned}$$

3) It is trivial by the definition of the Caldero-Chapoton map. \square

4. A \mathbb{Z} -BASIS FOR CYCLIC QUIVERS

In this section, we will focus on studying the following set

$$\mathcal{B}(Q) = \{X_R | \text{Ext}_{kQ}^1(R, R) = 0\}.$$

We prove that $\mathcal{B}(Q)$ is a \mathbb{Z} -basis of the algebra $\mathcal{AH}(Q)$ generated by all these generalized cluster variables. We first give the following definition.

Definition 4.1. For $M, N \in \text{mod } kQ$ with $\underline{\dim} M = (m_1, \dots, m_r)$ and $\underline{\dim} N = (s_1, \dots, s_r)$, we write $\underline{\dim} M \preceq \underline{\dim} N$ if $m_i \leq s_i$ for $1 \leq i \leq r$. Moreover, if there exists some i such that $m_i < s_i$, then we write $\underline{\dim} M \prec \underline{\dim} N$.

Remark 4.2. It is easy to see that $\underline{\dim} E_{i+2}[n-1] \prec \underline{\dim} E_i[n+1]$ and $\underline{\dim} E_i[n-1] \prec \underline{\dim} E_i[n+1]$ in Corollary 2.5.

Lemma 4.3. Let T_1, T_2 be kQ -modules such that $\underline{\dim} T_1 = \underline{\dim} T_2$. Then we have

$$X_{T_1} = X_{T_2} + \sum_{\underline{\dim} R \prec \underline{\dim} T_2} a_R X_R$$

where $R \in \text{mod } kQ$ and $a_R \in \mathbb{Z}$.

Proof. Suppose $T_1 = T_{11} \oplus T_{12} \oplus \dots \oplus T_{1m}$ and $\underline{\dim} T_1 = (d_1, d_2, \dots, d_r)$ where $T_{1i} (1 \leq i \leq m)$ are indecomposable regular modules with quasi-socle E_{i_1} and $\underline{\dim} T_{1i} = (d_{1i}, d_{2i}, \dots, d_{ri})$ for $1 \leq i \leq m$. Thus, we can see that $(d_1, d_2, \dots, d_r) = \sum_{i=1}^m (d_{1i}, d_{2i}, \dots, d_{ri})$. By Corollary 2.5 and Theorem 3.1, we have

$$\begin{aligned} & X_{E_1}^{d_1} X_{E_2}^{d_2} \dots X_{E_r}^{d_r} \\ &= \prod_{i=1}^m (X_{E_{i_1}} X_{E_{i_1+1}} X_{E_{i_1+2}} \dots X_{E_{i_1+d_{1i}+\dots+d_{ri}-1}}) \\ &= \prod_{i=1}^m (X_{T_{1i}} + \sum_{\underline{\dim} L' \prec \underline{\dim} T_{1i}} a_{L'} X_{L'}) \\ &= X_{T_1} + \sum_{\underline{\dim} L \prec \underline{\dim} T_1} a_L X_L. \end{aligned}$$

where $a_{L'}, a_L$ are integers. Similarly we have

$$X_{E_1}^{d_1} X_{E_2}^{d_2} \dots X_{E_r}^{d_r} = X_{T_2} + \sum_{\underline{\dim} M \prec \underline{\dim} T_2} b_M X_M$$

where b_M are integers.

Thus

$$X_{T_1} + \sum_{\underline{\dim} L' \prec \underline{\dim} T_1} a_{L'} X_{L'} = X_{T_2} + \sum_{\underline{\dim} M \prec \underline{\dim} T_2} b_M X_M.$$

Therefore, we have

$$X_{T_1} = X_{T_2} + \sum_{\underline{\dim} R \prec \underline{\dim} T_2} a_R X_R$$

where a_R are integers. □

We explain the method used in Lemma 4.3 by the following example.

Example 4.4. Consider $r = 4, X_{E_2[5]}$ and $X_{E_1[4] \oplus E_2}$. We can see that $\underline{\dim}(E_1[4] \oplus E_2) = \underline{\dim} E_2[5] = \underline{\dim}(E_1 \oplus 2E_2 \oplus E_3 \oplus E_4)$ and satisfy the conditions in Lemma

4.3. Thus, for $X_{E_1[4] \oplus E_2}$, we have

$$\begin{aligned}
X_{E_1} X_{E_2}^2 X_{E_3} X_{E_4} &= X_{E_1} X_{E_2} X_{E_3} X_{E_4} X_{E_2} \\
&= (X_{E_1[2]} + 1) X_{E_3} X_{E_4} X_{E_2} \\
&= (X_{E_1[3]} + X_{E_1}) X_{E_4} X_{E_2} + X_{E_3} X_{E_4} X_{E_2} \\
&= (X_{E_1[4]} + X_{E_1[2]}) X_{E_2} + X_{E_1} X_{E_4} X_{E_2} + X_{E_3} X_{E_4} X_{E_2} \\
&= X_{E_1[4] \oplus E_2} + X_{E_1[2] \oplus E_2} + (X_{E_1[2]} + 1) X_{E_4} + (X_{E_2[2]} + 1) X_{E_4} \\
&= X_{E_1[4] \oplus E_2} + X_{E_1[2] \oplus E_2} + X_{E_1[2] \oplus E_4} + X_{E_2[3]} + X_{E_2} + 2X_{E_4}.
\end{aligned}$$

Similarly for $X_{E_2[5]}$, we have

$$\begin{aligned}
X_{E_1} X_{E_2}^2 X_{E_3} X_{E_4} &= X_{E_2} X_{E_3} X_{E_4} X_{E_1} X_{E_2} \\
&= (X_{E_2[2]} + 1) X_{E_4} X_{E_1} X_{E_2} \\
&= (X_{E_2[3]} + X_{E_2}) X_{E_1} X_{E_2} + X_{E_4} X_{E_1} X_{E_2} \\
&= (X_{E_2[4]} + X_{E_2[2]}) X_{E_2} + X_{E_2} X_{E_1} X_{E_2} + X_{E_4} X_{E_1} X_{E_2} \\
&= X_{E_2[5]} + X_{E_2[3]} + X_{E_2[2] \oplus E_2} + (X_{E_1[2]} + 1) X_{E_2} + (X_{E_1[2]} + 1) X_{E_4} \\
&= X_{E_2[5]} + X_{E_2[3]} + X_{E_2[2] \oplus E_2} + X_{E_1[2] \oplus E_2} + X_{E_2} + X_{E_1[2] \oplus E_4} + X_{E_4}.
\end{aligned}$$

Hence, $X_{E_1[4] \oplus E_2} = X_{E_2[5]} + X_{E_2[2] \oplus E_2} - X_{E_4}$, where $\underline{\dim}(E_2[2] \oplus E_2) \prec \underline{\dim} E_2[5], \underline{\dim} E_4 \prec \underline{\dim} E_2[5]$.

Lemma 4.5.

$$X_{E_i[r]} = X_{E_{i+1}[r-2]} + 2.$$

Proof. According to Proposition 2.1. \square

Lemma 4.6. For any $M, N \in \text{mod } kQ$, $X_M X_N$ is a \mathbb{Z} -linear combination of the elements in $\mathcal{B}(Q)$.

Proof. By Theorem 3.1, we know that $X_M X_N$ must be a \mathbb{Z} -linear combination of elements in the set

$$\{X_{T \oplus R} | \text{Ext}_{kQ}^1(T, R) = \text{Ext}_{kQ}^1(R, T) = 0\}$$

where R is 0 or any regular exceptional module and T is 0 or any indecomposable regular module with self-extension.

By Lemma 4.3 and Lemma 4.5, we can easily find that $X_M X_N$ is actually a \mathbb{Z} -linear combination of elements in the set $\mathcal{B}(Q)$. \square

Proposition 4.7. Let $\Omega = \{A = (a_{ij}) \in M_{r \times r}(\mathbb{Z}_{\geq 0}) \mid a_{i,r} \cdots a_{i,r+i-2} \neq 0\}$ where $a_{i,r+s} = a_{i,s}$ for $i \geq 2$ and $s \in \mathbb{N}$. Let $E(A, i) = E_i^{a_{i,1}} \oplus \cdots \oplus E_{i+r-2}^{a_{i,r-1}}$ for $A \in \Omega$ and $i = 1, \dots, r$. Then the set $\mathcal{B}'(Q) = \{X_{E(A,i)} \mid i = 1, \dots, r, A \in \Omega\}$ is a linearly independent set over \mathbb{Z} .

Proof. Suppose that there exists the identity $S := \sum_{A \in \Omega_0, i=1, \dots, r} n(A, i) X_{E(A,i)} = 0$ where Ω_0 is a finite subset of Ω and $n(A, i) \neq 0 \in \mathbb{Z}$ for $i = 1, \dots, r$. Note that

$$X_{E(A,i)} = \prod_{j=i}^{i+r-2} \left(\frac{x_{j+1} + x_{j-1}}{x_j} \right)^{a_{i,j-i+1}}$$

for $i = 1, \dots, r$. Define a lexical order by set $x_r < x_1 < x_2 < \cdots < x_{r-1}$ and $x_i^a < x_i^b$ if $a < b$. Set $l_r(A) = \max\{a_{2,r-1}, \dots, a_{r,1}\}$ and $l_r = \max\{l_r(A)\}_{A \in \Omega_0}$. Then $l_r \neq 0$. Note that $a_{i,r-i+1}$ is just the exponent of X_{E_r} in the expression of $X_{E(A,i)}$ for $i = 2, \dots, r$. Then the expression of $\sum_{A \in \Omega_0, i=1, \dots, r} n(A, i) X_{E(A,i)}$ contains the unique part of the form $\frac{L(x_1, \dots, x_{r-1})}{x_r^{l_r}}$ which has the minimal exponent at x_r and $L(x_1, \dots, x_{r-1})$ is a Laurent polynomial associated to x_1, \dots, x_{r-1} . In

fact, $\frac{L(x_1, \dots, x_{r-1})}{x_r^{l_r}}$ is a part of the sum $\sum_{i=2, \dots, r, A \in \Omega_0; a_{i, r-i+1}=l_r} n(A, i) X_{E(A, i)}$. Note that the terms in this sum have a common factor $X_{E_r}^{l_r}$, thus we have the following identity

$$\sum_{i=2, \dots, r, A \in \Omega_0; a_{i, r-i+1}=l_r} n(A, i) X_{E(A, i)} = (*) X_{E_r}^{l_r}$$

here we denote $\frac{1}{X_{E_r}^{l_r}} \sum_{i=2, \dots, r, A \in \Omega_0; a_{i, r-i+1}=l_r} n(A, i) X_{E(A, i)}$ by $(*)$.

Now we set $l_{r+1}(A) = \max\{a_{3, r-1}, \dots, a_{r, 2}\}$ and $l_{r+1} = \max\{l_r(A)\}_{A \in \Omega_0}$. Then $l_{r+1} \neq 0$. In the same way as above, we know that the expression of the term $(*)$ contains the unique part of the form $\frac{L(x_2, \dots, x_{r-1})}{x_1^{l_{r+1}}}$ which has the minimal exponent at $x_1 = x_{r+1}$ and $L(x_2, \dots, x_{r-1})$ is a Laurent polynomial associated to x_2, \dots, x_{r-1} . Note that $\frac{L(x_2, \dots, x_{r-1})}{x_1^{l_{r+1}}} X_{E_r}^{l_r}$ is actually a part of the following term

$$\sum_{i=3, \dots, r, A \in \Omega_0; a_{i, r-i+1}=l_r, a_{i, r-i+2}=l_{r+1}} n(A, i) X_{E(A, i)} = (**) X_{E_1}^{l_{r+1}} X_{E_r}^{l_r}$$

here we denote $\frac{1}{X_{E_1}^{l_{r+1}} X_{E_r}^{l_r}} \sum_{i=3, \dots, r, A \in \Omega_0; a_{i, r-i+1}=l_r, a_{i, r-i+2}=l_{r+1}} n(A, i) X_{E(A, i)}$ by $(**)$.

Continue this discussion, we deduce that there exists some $n(A, i) = 0$. It is a contradiction. \square

Theorem 4.8. *The set $\mathcal{B}(Q)$ is a \mathbb{Z} -basis of the algebra $\mathcal{AH}(Q)$.*

Proof. It is easy to prove that the elements in $\mathcal{B}'(Q)$ and elements in $\mathcal{B}(Q)$ have a unipotent matrix transformation. Then by Proposition 4.7 and Lemma 4.6, we know that $\mathcal{B}(Q)$ is a \mathbb{Z} -basis of the algebra $\mathcal{AH}(Q)$. \square

Example 4.9. (1) Consider $r = 1$, then we can calculate

$$X_{E_1} = 2, X_{E_1[2]} = 3, X_{E_1[3]} = 4, \dots, X_{E_1[n]} = n + 1, \dots$$

It is obvious that $\mathcal{B}(Q) = \{1\}$.

(2) Consider $r = 2$, then we can calculate

$$\begin{aligned} X_{E_1} &= \frac{2x_2}{x_1}, X_{E_1[2]} = 3, \dots, X_{E_1[2n-1]} = \frac{2nx_2}{x_1}, X_{E_1[2n]} = 2n + 1, \dots \\ X_{E_2} &= \frac{2x_1}{x_2}, X_{E_2[2]} = 3, \dots, X_{E_2[2n-1]} = \frac{2nx_1}{x_2}, X_{E_2[2n]} = 2n + 1, \dots \end{aligned}$$

It is obvious that $\mathcal{B}(Q) = \{X_{E_1}^m, X_{E_2}^n | m, n \in \mathbb{Z}_{\geq 0}\}$.

ACKNOWLEDGEMENTS

The authors are grateful to Professor Jie Xiao for helpful discussions.

REFERENCES

- [BMRRT] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*. Advances in Math. 204 (2006), 572-618.
- [CC] P. Caldero and F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*. Comm. Math. Helvetici, 81 (2006), 596-616.
- [CK] P. Caldero and B. Keller, *From triangulated categories to cluster algebras*. Invent. math. 172 (2008), no. 1, 169-211.
- [CK2] P. Caldero and B. Keller, *From triangulated categories to cluster algebras II*. Annales Scientifiques de l'Ecole Normale Supérieure, 39 (4) (2006), 83-100.
- [CZ] P. Caldero and A. Zelevinsky, *Laurent expansions in cluster algebras via quiver representations*. Moscow Math. J. 6 (2006), no. 2, 411-429.
- [DX] M. Ding and F. Xu, *A \mathbb{Z} -basis for the cluster algebra of type \tilde{D}_4* , to appear in Algebra Colloquium.
- [DXX] M. Ding, J. Xiao and F. Xu, *Integral bases of cluster algebras and representations of tame quivers*. arXiv:0901.1937v1 [math.RT].

- [FZ] S. Fomin and A. Zelevinsky, *Cluster algebras. I. Foundations*. J. Amer. Math. Soc. 15 (2002), no. 2, 497-529.
- [GLS] C. Geiss, B. Leclerc, and J. Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*. ArXiv:math/0703039v2, 2008.
- [Ke] B. Keller, *On triangulated orbit categories*. Documenta Math. 10 (2005), 551-581.
- [MRZ] R. Marsh, M. Reineke, and A. Zelevinsky, *Generalized associahedra via quiver representations*. Trans. A.M.S, 355(1) (2003), 4171-4186.
- [Pa] Y. Palu, *Cluster characters for 2-Calabi-Yau triangulated categories*. Annales de l'institut Fourier, 58 no. 6 (2008), 2221-2248.
- [Ro] Adam-Christiaan van Roosmalen, *Abelian 1-Calabi-Yau categories*. Int. Math. Res. Not. IMRN (2008), no. 6, Art. ID rnn003, 20.
- [XX] J. Xiao and F. Xu, *Green's formula with \mathbb{C}^* -action and Caldero-Keller's formula*. arXiv:0707.1175. To appear in Prog. Math.

INSTITUTE FOR ADVANCED STUDY, TSINGHUA UNIVERSITY, BEIJING 100084, P. R. CHINA

E-mail address: m-ding04@mails.tsinghua.edu.cn (M.Ding)

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, P. R. CHINA

E-mail address: fanxu@mail.tsinghua.edu.cn (F.Xu)